補足 6－1 フィッシャー情報量の計算

能力θの尤度関数が次式

$$L(\theta)=\prod_{i=1}^{n}P(\theta;a_i,b_i,c_i)^{u_i}Q(\theta;a_i,b_i,c_i)^{1-u_i}$$

で与えられるとき、その対数尤度の１次導関数は次式で与えられる。

$$\frac{d}{d\theta}\log L(\theta)=\sum_{i=1}^{n}\left\{u_i\frac{P'(\theta;a_i,b_i,c_i)}{P(\theta;a_i,b_i,c_i)}+(1-u_i)\frac{Q'(\theta;a_i,b_i,c_i)}{Q(\theta;a_i,b_i,c_i)}\right\}$$

したがって、２次導関数は次式となる。

$$\frac{d^2}{d\theta^2}\log L(\theta)=\sum_{i=1}^{n}\left\{u_i\frac{P''(\theta;a_i,b_i,c_i)P(\theta;a_i,b_i,c_i)-P'(\theta;a_i,b_i,c_i)^2}{P^2(\theta;a_i,b_i,c_i)}\right.$$
$$\left.+(1-u_i)\frac{Q''(\theta;a_i,b_i,c_i)Q(\theta;a_i,b_i,c_i)-Q'(\theta;a_i,b_i,c_i)^2}{Q^2(\theta;a_i,b_i,c_i)}\right\}$$

上式の期待値は、次式で与えられる。

$$E\left[\frac{d^2}{d\theta^2}\log L(\theta)\right]$$
$$=\sum_{i=1}^{n}\left\{E(u_i)\cdot\frac{P''(\theta;a_i,b_i,c_i)P(\theta;a_i,b_i,c_i)-P'(\theta;a_i,b_i,c_i)^2}{P^2(\theta;a_i,b_i,c_i)}\right.$$
$$\left.+E(1-u_i)\cdot\frac{Q''(\theta;a_i,b_i,c_i)Q(\theta;a_i,b_i,c_i)-Q'(\theta;a_i,b_i,c_i)^2}{Q^2(\theta;a_i,b_i,c_i)}\right\}$$

$$=\sum_{i=1}^{n}\left\{P(\theta;a_i,b_i,c_i)\cdot\frac{P''(\theta;a_i,b_i,c_i)P(\theta;a_i,b_i,c_i)-P'(\theta;a_i,b_i,c_i)^2}{P^2(\theta;a_i,b_i,c_i)}\right.$$
$$\left.+Q(\theta;a_i,b_i,c_i)\cdot\frac{Q''(\theta;a_i,b_i,c_i)Q(\theta;a_i,b_i,c_i)-Q'(\theta;a_i,b_i,c_i)^2}{Q^2(\theta;a_i,b_i,c_i)}\right\}$$

$$=\sum_{i=1}^{n}\left[\frac{1}{P(\theta;a_i,b_i,c_i)Q(\theta;a_i,b_i,c_i)}\times\right.$$
$$\left\{\left(P''(\theta;a_i,b_i,c_i)P(\theta;a_i,b_i,c_i)-P'(\theta;a_i,b_i,c_i)^2\right)Q(\theta;a_i,b_i,c_i)\right.$$
$$\left.+\left(Q''(\theta;a_i,b_i,c_i)Q(\theta;a_i,b_i,c_i)-Q'(\theta;a_i,b_i,c_i)^2\right)P(\theta;a_i,b_i,c_i)\right\}$$

ここで、

$$Q'(\theta;a_i,b_i,c_i)=-P'(\theta;a_i,b_i,c_i)$$

$$Q''(\theta;a_i,b_i,c_i)=-P''(\theta;a_i,b_i,c_i)$$

に注意して、上式を変形すると次式を得る。

$$E\left[\frac{d^2}{d\theta^2}\log L(\theta)\right]$$

$$=\sum_{i=1}^{n}\left[\frac{1}{P(\theta;a_i,b_i,c_i)Q(\theta;a_i,b_i,c_i)}\times\left\{-P'(\theta;a_i,b_i,c_i)^2Q(\theta;a_i,b_i,c_i)-Q'(\theta;a_i,b_i,c_i)^2P(\theta;a_i,b_i,c_i)\right\}\right]$$

$$=\sum_{i=1}^{n}\frac{-P'(\theta;a_i,b_i,c_i)^2\{Q(\theta;a_i,b_i,c_i)+P(\theta;a_i,b_i,c_i)\}}{P(\theta;a_i,b_i,c_i)Q(\theta;a_i,b_i,c_i)}$$

$$=\sum_{i=1}^{n}\frac{-P'(\theta;a_i,b_i,c_i)^2}{P(\theta;a_i,b_i,c_i)Q(\theta;a_i,b_i,c_i)}$$

以上より、フィッシャー情報量$I(\theta)$、すなわちテスト情報関数は、次式で与えられることがわかる。

$$I(\theta)=-E\left[\frac{d^2}{d\theta^2}\log L(\theta)\right]=\sum_{i=1}^{n}\frac{P'(\theta;a_i,b_i,c_i)^2}{P(\theta;a_i,b_i,c_i)Q(\theta;a_i,b_i,c_i)}$$